International Journal of Theoretical Physics, Vol. 40, No. 2, 2001

Gauged Nonlinear Sigma Model in the Instant Form: Hamiltonian and BRST Formulations

Usha Kulshreshtha[1]

Received April 26, 2000

A gauged nonlinear sigma model in one space, one time dimension is considered in the usual instant form of dynamics on the hyperplanes x^0 = const. The theory is seen to possess a local vector gauge symmetry. The Hamiltonian and BRST formulations of this theory are investigated with specific gauge choices.

1. INTRODUCTION

The $O(N)$ nonlinear sigma models (NLSM) in one space, one time (1 + 1) dimension [1–13], where the sigma field is a real N-component field, provide a laboratory for the various nonperturbative techniques, e.g., the 1/N expansion [5–8], operator product expansion, and low-energy theorems [9, 10]. These models are characterized by features such as renormalization and asymptotic freedom common to quantum chromodynamics, and they exhibit a nonperturbative particle spectrum, have no intrinsic scale parameter, possess topological charges, and are very crucial in the context of conformal [5–8] and string field theories [11, 12], where they appear in the classical limit [9, 10].

The Hamitonian formulation of the gauge-noninvariant (GNI), $O(N)$ NLSM in (1 + 1) dimension has been studied in ref. 1 and its two gauge-invariant (GI) versions have been constructed in ref. 4, where the Hamiltonian [14, 15] and Recchi–Rouet–Stora and Tyutin (BRST) [16–27] quantization of these GI models have also been studied in detail [4]. The NLSM studied in refs. 1–4 do not have any gauge fields in the theory Corresponding to these models, if we consider the models involving the gauge field, as proposed

[1] Fachbereich Physik der Universität Kaiserslautern, Germany, and Department of Physics, University of Delhi, Delhi, India; e-mail: usha@physik.uni-kl.de, usha@physics.du.ac.in

0020-7748/01/0200-0491$19.50/0

in the present work, we obtain the so-called gauged-NLSM (GNLSM). In the present work, we propose to study such a GNLSM obtained by gauging the usual NLSM [without involving the vector gauge fields $A^{\mu}(x, t)$] [1–4] and investigate its canonical structure, constrained dynamics, and Hamiltonian [14, 15] and BRST [16–27] formulations in the usual instant form (IF) of dynamics on the hyperplanes $x^0 = \text{const}$ [28].

The model is seen to possess a set of five first-class constraints (two primary and three secondary), implying that the theory under consideration is a GI theory. The Hamiltonian formulation of this GNLSM is investigated under specific gauge-fixing conditions.

However, in the usual Hamiltonian formulation of a GI theory under gauge-fixing conditions, one necessarily destroys the gauge invariance of the theory by fixing the gauge (which converts a set of first-class constraints into a set of second-class constraints, implying a breaking of gauge invariance under the gauge fixing). To achieve the quantization of a GI theory such that the gauge invariance of the theory is maintained even under the gauge fixing, one goes to a more generalized procedure called the BRST formulation [4, 16–27]. In the BRST formulation of a GI theory, the theory is rewritten as a quantum system that possesses a generalized gauge invariance called the BRST symmetry. For this, one enlarges the Hilbert space of the GI theory and replaces the notion of the gauge transformation, which shifts operators by c-number functions, by a BRST transformation, which mixes the operators having different statistics. In view of this, one introduces new anticommuting variables c and $\bar{c}$ called the Faddeev–Popov ghost and anti-ghost fields, respectively, which are Grassmann numbers on the classical level and operators in the quantized theory, and a commuting variable b called the Nakanishi–Lautrup field [3, 16–27].

In the BRST formulation of a theory, one thus embeds a GI theory into a BRST-invariant system, and the quantum Hamiltonian of the system (which includes the gauge-fixing contribution) commutes with the BRST charge operator Q as well as with the anti-BRST charge operator $\overline{Q}$. The new symmetry of the system (the BRST symmetry) that replaces the gauge invariance is maintained (even under gauge fixing), and hence projecting any state onto the sector of BRST and anti-BRST invariant states yields a theory which is isomorphic to the original GI theory. The unitarity and consistency of the BRST-invariant theory described by the gauge-fixed quantum Lagrangian is guaranteed by the conservation and nilpotency of the BRST charge Q.

In the next section, we briefly recapitulate the basics of the usual $O(N)$-NLSM (without gauge fields) [1–4]. In Section 3, we study the Hamiltonian formulation of the proposed GNLSM, and in Section 4, its BRST formulation under specific gauge-fixing conditions. The summary and discussion is given in Section 5.

2. A RECAPITULATION OF THE *O*(*N*)-NLSM

The $O(N)$-nonlinear sigma model in one space, one time dimension is described by the Lagrangian density [1–13]

$$\mathcal{L}^N = \left[\frac{1}{2}\,\partial_\mu\sigma_k\partial^\mu\sigma_k + \lambda(\sigma_k^2 - 1)\right], \qquad k = 1, 2, \ldots, N \tag{2.1a}$$

$$= \left[\frac{1}{2}\,(\dot{\sigma}_k^2 - \sigma_k'^2) + \lambda(\sigma_k^2 - 1)\right], \qquad k = 1, 2, \ldots, N \tag{2.1b}$$

Here $\vec{\sigma} \equiv [\sigma_k(x, t); k = 1,2, \ldots, N]$ is a multiplet of N real scalar fields in $(1 + 1)$ dimension and $\lambda(x, t)$ is another scalar field. Overdot and prime denote time and space derivatives, respectively. The field $\vec{\sigma}(x, t)$ maps the two-dimensional space-time into the N-dimensional internal manifold whose coordinates are $\sigma_k(x, t)$. This model is seen to possess a set of four second-class constraints [1–4]

$$\rho_1 = p_\lambda \approx 0 \tag{2.2a}$$

$$\rho_2 = [\sigma_k^2 - 1] \approx 0 \tag{2.2b}$$

$$\rho_3 = 2\sigma_k\,\Pi_k \approx 0 \tag{2.2c}$$

$$\rho_4 = (2\Pi_k^2 + 4\lambda\sigma_k^2 + 2\sigma_k\sigma_k'') \approx 0 \tag{2.2d}$$

where ρ_1 is a primary constraint and ρ_2, ρ_3, and ρ_4 are secondary constraints. Here Π_k and p_λ are the momenta canonically conjugate, respectively, to σ_k and λ. The nonvanishing equal-time Dirac brackets (DBs) of the theory are given by [1, 4]

$$\{\Pi_l(x), \Pi_m(y)\}_{\mathrm{D}} = \frac{-1}{\sigma_k^2}\,[\sigma_l(x)\Pi_m(y) - \Pi_l(x)\sigma_m(y)]\delta(x - y) \tag{2.3a}$$

$$\{\sigma_l(x), \Pi_m(y)\}_{\mathrm{D}} = \left[\delta_{lm} - \frac{\sigma_l(x)\sigma_m(y)}{\sigma_k^2}\right]\delta(x - y) \tag{2.3b}$$

In achieving the canonical quantization of the theory, one encounters the problem of operator ordering while going from DBs to the commutation relations. This problem can be resolved, as explained in refs. 1, 2, and 4, by demanding that all the fields and field momenta after quantization become Hermitian operators and that all the canonical commutation relations be consistent with the hermiticity of these operators [1, 2, 4].

3. THE GAUGED NONLINEAR SIGMA MODEL (GNLSM)

The $O(N)$-GNLSM that we propose to study in the present work is described by the Lagrangian density in $(1 + 1)$ dimension

$$\mathcal{L} := [\tfrac{1}{2}\partial_\mu\sigma_k\partial^\mu\sigma_k + \lambda(\sigma_k^2 - 1) - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - eA_\mu\,\partial^\mu\sigma_k + \tfrac{1}{2}e^2\,A_\mu A^\mu] \tag{3.1a}$$

$$= [\tfrac{1}{2}(\mathring{\sigma}_k^2 - \sigma_k'^2) + \lambda(\sigma_k^2 - 1) + \tfrac{1}{2}(\mathring{A}_1 - A_0')^2 - e(A_0\mathring{\sigma}_k - A_1\sigma_k') + \tfrac{1}{2}e^2(A_0^2 - A_1^2)] \tag{3.1b}$$

$$F^{\mu\nu} = (\partial^\mu A^\nu - \partial^\nu A^\mu); \qquad g^{\mu\nu} := \mathrm{diag}(+1, -1) \tag{3.1c}$$

In Eq. (3.1b), the first term corresponds to a massless boson (which is equivalent to a massless fermion), the second term is the usual term involving the nonlinear constraint ($\sigma_k^2 - 1 \approx 0$) and the auxiliary field λ, the third term is the kinetic energy term of the electromagnetic vector-gauge field $A_\mu(x, t)$, the fourth term represents the coupling of the sigma field to the electromagnetic field, and the last term is the mass term for the vector gauge boson $A_\mu(x, t)$. The Euler–Lagrange equations obtained from $\mathcal{L}$ are

$$[\mathring{\mathring{\sigma}}_k - \sigma_k''] = [2\lambda\sigma_k + e(\mathring{A}_0 - A_1')] \tag{3.2a}$$

$$[\mathring{A}_1' - A_0''] = [e\mathring{\sigma}_k - e^2A_0] \tag{3.2b}$$

$$[\mathring{\mathring{A}}_1 - \mathring{A}_0'] = [e\sigma_k' - e^2A_1] \tag{3.2c}$$

$$[\sigma_k^2 - 1] = 0 \tag{3.2d}$$

The canonical momenta for the above GNLSM obtained from $\mathcal{L}$ are

$$\Pi_k := \partial\mathcal{L}/\partial\mathring{\sigma}_k = [\mathring{\sigma}_k - eA_0] \tag{3.3a}$$

$$p_\lambda := \partial\mathcal{L}/\partial\mathring{\lambda} = 0 \tag{3.3b}$$

$$\Pi_0 := \frac{\partial\mathcal{L}}{\partial\mathring{A}_0} = 0 \tag{3.3c}$$

$$E(:= \Pi^1) := \frac{\partial\mathcal{L}}{\partial\mathring{A}_1} = [\mathring{A}_1 - A_0'] \tag{3.3d}$$

Equations (3.3b) and (3.3c) imply that $\mathcal{L}$ possesses two primary constraints:

$$\Omega_1 = \Pi_0 \approx 0 \tag{3.4a}$$

$$\Omega_2 = p_\lambda \approx 0 \tag{3.4b}$$

The canonical Hamiltonian density corresponding to $\mathcal{L}$ is

$$\begin{aligned}\mathcal{H}_C &= [\Pi_k\mathring{\sigma}_k + p_\lambda\mathring{\lambda} + \Pi_0\mathring{A}_0 + E\mathring{A}_1 - \mathcal{L}] \\ &= [\tfrac{1}{2}(\Pi_k^2 + E^2 + \sigma_k'^2 + e^2A_0^2) + EA_0' + eA_0\Pi_k - eA_1\sigma_k' \\ &\quad - \lambda(\sigma_k^2 - 1) - \tfrac{1}{2}e^2\,(A_0^2 - A_1^2)]\end{aligned} \tag{3.5}$$

After including the primary constraints Ω_1 and Ω_2 in the canonical Hamiltonian density $\mathcal{H}_c$, with the help of Lagrange multipliers u and v, one can write the total Hamiltonian density $\mathcal{H}_T$ as

$$\mathcal{H}_T = [\mathcal{H}_c + \Pi_0 u + p_\lambda v] \tag{3.6}$$

The Hamilton equations obtained from the total Hamiltonian $H_T = \int \mathcal{H}_T\, dx$ are

$$\mathring{\sigma}_k = \partial H_T/\partial \Pi_k = [\Pi_k + eA_0] \tag{3.7a}$$

$$-\mathring{\Pi}_k = \partial H_T/\partial \sigma_k = [eA_1' - 2\lambda\sigma_k - \sigma_k''] \tag{3.7b}$$

$$\mathring{\lambda} = \partial H_T/\partial p_\lambda = v \tag{3.7c}$$

$$-\mathring{p}_\lambda = \partial H_T/\partial \lambda = [\sigma_k^2 - 1] \tag{3.7d}$$

$$\mathring{A}_0 = \partial H_T/\partial \Pi_0 = u \tag{3.7e}$$

$$-\mathring{\Pi}_0 = \partial H_T/\partial A_0 = [e\Pi_k - E'] \tag{3.7f}$$

$$\mathring{A}_1 = \partial H_T/\partial E = [E + A_0'] \tag{3.7g}$$

$$-\mathring{E} = \partial H_T/\partial A_1 = [e^2 A_1 - e\sigma_k'] \tag{3.7h}$$

$$\mathring{u} = \partial H_T/\partial \Pi_u = 0 \tag{3.7i}$$

$$-\mathring{\Pi}_u = \partial H_T/\partial u = \Pi_0 \tag{3.7j}$$

$$\mathring{v} = \partial H_T/\partial \Pi_v = 0 \tag{3.7k}$$

$$-\mathring{\Pi}_v = \partial H_T/\partial v = p_\lambda \tag{3.7l}$$

These are the equations of motion that preserve the constraints of the theory, Ω_1 and Ω_2, in the course of time. Here Π_u and Π_v are the momenta canonically conjugate, respectively, to u and v. For the Poisson bracket $\{\ ,\ \}_P$ of two functions A and B, we choose the convention

$$\{A(x), B(y)\}_P := \int dz \sum_\alpha \left[\frac{\partial A(x)}{\partial q_\alpha(z)}\frac{\partial B(y)}{\partial p_\alpha(z)} - \frac{\partial A(x)}{\partial p_\alpha(z)}\frac{\partial B(y)}{\partial q_\alpha(z)}\right] \tag{3.8}$$

Demanding that primary constraint Ω_1 be preserved in the course of time, we obtain the secondary constraint

$$\Omega_3 := \{\Omega_1, \mathcal{H}_T\}_P = [E' - e\Pi_k] \approx 0 \tag{3.9}$$

Demanding the preservation of Ω_3 with time does not lead to any further constraints. Demanding the preservation of Ω_2 with time, however, leads to a secondary constraint

$$\Omega_4 := \{\Omega_2, \mathcal{H}_T\}_P = [\sigma_k^2 - 1] \approx 0 \tag{3.10}$$

and this constraint in turn leads to a further constraint:

$$\Omega_5 := \{\Omega_4, \mathscr{H}_T\}_P = [2\sigma_k\Pi_k + 2eA_0\sigma_k] \approx 0 \tag{3.11}$$

The preservation of Ω_5 for all time does not give rise to any further constraints. The theory is thus seen to possess five constraints Ω_1, Ω_2, Ω_3, Ω_4, and Ω_5:

$$\Omega_1 = \Pi_0 \approx 0 \tag{3.12a}$$

$$\Omega_2 = p_\lambda \approx 0 \tag{3.12b}$$

$$\Omega_3 = [E' - e\Pi_k] \approx 0 \tag{3.12c}$$

$$\Omega_4 = [\sigma_k^2 - 1] \approx 0 \tag{3.12d}$$

$$\Omega_5 = [2\sigma_k\Pi_k + 2eA_0\sigma_k] \approx 0 \tag{3.12e}$$

The matrix of the Poisson brackets of the constraints Ω_i, namely $M_{\alpha\beta}(z, z')$ $:= \{\Omega_\alpha(z), \Omega_\beta(z')\}_P$, is then calculated. The nonvanishing matrix elements of the matrix $M_{\alpha\beta}(z, z')$ (with the arguments of the field variables being suppressed) are

$$M_{15} = -M_{51} = -2e\sigma_k\delta(z - z') \tag{3.13a}$$

$$M_{34} = -M_{43} = 2e\sigma_k\delta(z - z') \tag{3.13b}$$

$$M_{35} = -M_{53} = 2e(\Pi_k + eA_0)\delta(z - z') \tag{3.13c}$$

$$M_{45} = -M_{54} = 4\sigma_k^2\delta(z - z') \tag{3.13d}$$

The inverse of the matrix $M_{\alpha\beta}$ does not exist and therefore the matrix is singular, implying that the set of constraints Ω_i is first class and that the theory described by $\mathscr{L}$ is a GI theory [4]. In fact, the action of theory $S = \int \mathscr{L}\, dx\, dt$ is seen to be invariant under the local vector gauge transformation (LVGT):

$$\delta\sigma_k = e\beta(x, t), \qquad \delta A_1 = \beta'(x, t), \qquad \delta A_0 = \mathring{\beta}(x, t) \tag{3.14a}$$

$$\delta\lambda = -\mathring{\beta}(x, t), \qquad \delta\Pi_k = \delta E = \delta\Pi_0 = \delta p_\lambda = \delta\Pi_u = \delta\Pi_v = 0 \tag{3.14b}$$

$$\delta u = \partial_0\partial_0\beta(x, t), \qquad \delta v = -\partial_0\partial_0\beta(x, t) \tag{3.14c}$$

where $\beta \equiv \beta(x, t)$ is an arbitrary function of its arguments. The generator of the LVGT is the charge operator of the theory:

$$J^0 = \int j^0\, dx = \int dx\, [e\beta(\mathring{\sigma}_k - eA_0) + \beta'(\mathring{A}_1 - A_0')] \tag{3.15}$$

the current operator of the theory is

$$J^1 = \int j^1\, dx = \int dx\, [e\beta(-\sigma_k' + eA_1) - \mathring{\beta}(\mathring{A}_1 - A_0')] \tag{3.16}$$

The divergence of the vector-current density, namely, $\partial_\mu j^\mu$, is therefore seen to vanish under the gauge constraint $\lambda \approx 0$. This implies that the theory possesses, at the classical level, a local vector gauge symmetry (LVGS) under the gauge constraint $\lambda \approx 0$. In the present work, we work under the gauge $\lambda \approx 0$, which is, in fact, equivalent to a temporal or time-axial kind of a gauge for the coordinate λ. We now proceed to quantize the theory under the gauge $\mathcal{G} = \lambda = 0$. Under this gauge, the total set of constraints of the theory becomes

$$\chi_1 = \Omega_1 = \Pi_0 \approx 0 \tag{3.17a}$$

$$\chi_2 = \Omega_2 = p_\lambda \approx 0 \tag{3.17b}$$

$$\chi_3 = \Omega_3 = [E' - e\Pi_k] \approx 0 \tag{3.17c}$$

$$\chi_4 = \Omega_4 = [\sigma_k^2 - 1] \approx 0 \tag{3.17d}$$

$$\chi_5 = \Omega_5 = [2\sigma_k\Pi_k + 2eA_0\sigma_k] \approx 0 \tag{3.17e}$$

$$\chi_6 = \mathcal{G} = \lambda \approx 0 \tag{3.17f}$$

The matrix of the Poisson brackets of the constraints χ_i namely, $R_{\alpha\beta}(z, z')$ $:= \{\chi_\alpha(z), \chi_\beta(z')\}_P$, is then calculated. The nonvanishing matrix elements of the matrix $R_{\alpha\beta}(z, z')$ (with the arguments of the field variables being suppressed again) are

$$R_{13} = -R_{31} = M_{13} \tag{3.18a}$$

$$R_{15} = -R_{51} = M_{15} \tag{3.18b}$$

$$R_{34} = -R_{43} = M_{34} \tag{3.18c}$$

$$R_{35} = -R_{53} = M_{35} \tag{3.18d}$$

$$R_{45} = -R_{54} = M_{45} \tag{3.18e}$$

$$R_{26} = -R_{62} = -\delta(z - z') \tag{3.18f}$$

The inverse of the matrix $R_{\alpha\beta}(z, z')$ exists and the matrix is nonsingular. The nonvanishing elements of the inverse of the matrix $R_{\alpha\beta}(z, z')$ [i.e., the elements of the matrix $(R^{-1})_{\alpha\beta}$ (with the arguments of the field variables being suppressed once again)] are

$$(R^{-1})_{13} = -(R^{-1})_{31} = \left[\frac{1}{e^2}\right]\delta(z - z') \tag{3.19a}$$

$$(R^{-1})_{14} = -(R^{-1})_{41} = \left[\frac{-\Pi_k - eA_0}{2e\sigma_k^2}\right]\delta(z - z') \tag{3.19b}$$

$$(R^{-1})_{15} = -(R^{-1})_{51} = \left[\frac{1}{2e\sigma_k}\right]\delta(z - z') \tag{3.19c}$$

$$(R^{-1})_{26} = -(R^{-1})_{62} = \delta(z - z') \tag{3.19d}$$

$$(R^{-1})_{34} = -(R^{-1})_{43} = \left[\frac{-1}{2e\sigma_k}\right]\delta(z - z') \tag{3.19e}$$

with

$$\int dz\, R(x, z)R^{-1}(z, y) = 1_{6\times 6}\, \delta(x - y) \tag{3.20}$$

The Dirac bracket $\{\cdot, \cdot\}_D$ of two functions A and B is defined as [14, 15]

$$\{A, B\}_D := \{A, B\}_P - \iint dz\, dz' \times \sum_{\alpha,\beta} [\{A, \Gamma_\alpha(z)\}_P[\Delta^{-1}_{\alpha\beta}(z, z')]\{\Gamma_\beta(z'), B\}_P] \tag{3.21}$$

where Γ_i are the constraints of the theory and $\Delta_{\alpha\beta}(z, z')$ [$:= \{\Gamma_\alpha(z), \Gamma_\beta(z')\}_P$] is the matrix of the Poisson brackets of the constraints Γ_i. The transition to quantum theory is made by the replacement of the Dirac brackets by the operator commutation relations according to

$$\{A, B\}_D \to (-i)[A, B]; \qquad i = \sqrt{-1} \tag{3.22}$$

The nonvanishing equal-time commutators of the theory described by $\mathscr{L}$ under the gauge $\mathscr{G} = \lambda = 0$ are finally obtained as

$$[A_0(x), \Pi_0(y)] = \left[\frac{-i}{e^2}\right]\delta(x - y) \tag{3.23a}$$

$$[A_1(x), \Pi_k(y)] = \left[\frac{-i}{e}\right]\delta'(x - y) \tag{3.23b}$$

$$[A_1(x), E(y)] = i\delta(x - y) \tag{3.23c}$$

$$[A_0(x), A_1(y)] = \left[\frac{i}{e^2}\right]\delta'(x - y) \tag{3.23d}$$

$$[\Pi_0(x), \Pi_k(y)] = \left[\frac{-i}{e}\right]\delta(x - y) \tag{3.23e}$$

For use in the next section, for considering the BRST formulation of our GI theory described by $\mathscr{L}$, we convert the total Hamiltonian density $\mathscr{H}_T$ into the first-order Lagrangian density:

$$\mathscr{L}_{10} := [\Pi_k \mathring{\sigma}_k + p_\lambda \mathring{\lambda} + \Pi_0 \mathring{A}_0 + E\mathring{A}_1 + \Pi_u \mathring{u} + \Pi_v \mathring{v} - \mathscr{H}_T] \tag{3.24a}$$

$$= [\Pi_k \mathring{\sigma}_k + E\mathring{A}_1 + \Pi_u \mathring{u} + \Pi_v \mathring{v} - \tfrac{1}{2}(\Pi_k^2 + E^2 + \sigma_k'^2 + e^2 A_0^2)$$
$$+ \lambda(\sigma_k^2 - 1) - EA_0' - eA_0\Pi_k + eA_1\sigma_k' + \tfrac{1}{2}e^2(A_0^2 - A_1^2)] \tag{3.24b}$$

4. BRST FORMULATION

4.1. BRST Invariance

For the BRST formulation of the GNLSM, we now rewrite our theory, which is GI, as a quantum system that possesses the generalized gauge invariance called the BRST symmetry. We first enlarge the Hilbert space of our gauge-invariant GNLSM and replace the notion of gauge transformation, which shifts operators by c-number functions, by a BRST transformation, which mixes operators with Bose and Fermi statistics. We then introduce new anticommuting variables c and $\bar{c}$ (Grassmann numbers on the classical level, operators in the quantized theory) and a commuting variable b (called the Nakanishi–Lautrup field) such that [3, 16–27]

$$\hat{\delta}\sigma_k = ec; \quad \hat{\delta}A_1 = c'; \quad \hat{\delta}A_0 = \mathring{c}; \quad \hat{\delta}\lambda = -\mathring{c} \tag{4.1a}$$

$$\hat{\delta}\Pi_k = \hat{\delta}E = \hat{\delta}\Pi_0 = \hat{\delta}p_\lambda = 0; \quad \hat{\delta}u = \partial_0\partial_0 c; \quad \hat{\delta}v = -\partial_0\partial_0 c \tag{4.1b}$$

$$\hat{\delta}\Pi_u = 0; \quad \hat{\delta}\Pi_v = 0 \tag{4.1c}$$

$$\hat{\delta}c = 0; \quad \hat{\delta}\bar{c} = b; \quad \hat{\delta}b = 0 \tag{4.1d}$$

with the property $\hat{\delta}^2 = 0$. We now define a BRST-invariant function of the dynamical variables to be a function $f(\Pi_k, p_\lambda, E, \Pi_0, \Pi_u, \Pi_v, p_b, \Pi_c, \Pi_{\bar{c}}, \sigma_k, \lambda, A_1, A_0, u, v, b, c, \bar{c})$ such that $\hat{\delta}f = 0$.

4.2 Gauge Fixing in the BRST Formalism

Gauge fixing in the BRST formalism implies adding to the first-order Lagrangian density $\mathscr{L}_{10}$ a trivial BRST-invariant function [3, 16–27]. We thus write

$$\mathscr{L}_{\text{BRST}} = \{\Pi_k \mathring{\sigma}_k + E\mathring{A}_1 + \Pi_u \mathring{u} + \Pi_v \mathring{v} - \tfrac{1}{2}(\Pi_k^2 + E^2 + \sigma_k'^2 + e^2 A_0^2)$$
$$+ \lambda(\sigma_k^2 - 1) - EA_0' - eA_0\Pi_k + eA_1\sigma_k' + \tfrac{1}{2}e^2(A_0^2 - A_1^2)$$
$$+ \hat{\delta}[\bar{c}(\mathring{A}_0 + \tfrac{1}{e}\sigma_k + \tfrac{1}{2}b)]\} \tag{4.2}$$

The last term in the above equation is the extra BRST-invariant gauge-fixing term. After one integration by parts, we write the above equation as

$$\mathcal{L}_{\text{BRST}} = [\Pi_k\mathring{\sigma}_k + E\mathring{A}_1 + \Pi_u\mathring{u} + \Pi_v\mathring{v} - \tfrac{1}{2}(\Pi_k^2 + E^2 + \sigma_k'^2 + e^2A_0^2) + \lambda(\sigma_k^2 - 1) - EA_0' - eA_0\Pi_k + eA_1\sigma_k' + \tfrac{1}{2}e^2(A_0^2 - A_1^2) + b(\mathring{A}_0 + \tfrac{1}{e}\sigma_k) + \tfrac{1}{2}b^2 + \mathring{\bar{c}}\mathring{c} - \bar{c}c] \tag{4.3}$$

Proceeding classically, we find for the Euler–Lagrange equation for b

$$-b = \left(\mathring{A}_0 + \frac{1}{e}\,\sigma_k\right) \tag{4.4}$$

The requirement $\hat{\delta}b = 0$ then implies

$$-\hat{\delta}b = \left(\hat{\delta}\mathring{A}_0 + \frac{1}{e}\,\hat{\delta}\sigma_k\right) \tag{4.5}$$

which in turn implies

$$-\partial_0\partial_0 c = c \tag{4.6}$$

The above equation is also an Euler–Lagrange equation obtained by the variation of $\mathcal{L}_{\text{BRST}}$ with respect to $\bar{c}$. In introducing momenta, one has to be careful in defining those for the fermionic variables. We thus define the bosonic momenta in the usual manner so that

$$\Pi_0 := \frac{\partial}{\partial(\partial_0 A_0)}\,\mathcal{L}_{\text{BRST}} = b \tag{4.7}$$

but for the fermionic momenta with directional derivatives, we set

$$\Pi_c := \mathcal{L}_{\text{BRST}}\,\frac{\overleftarrow{\partial}}{\partial(\partial_0 c)} = \mathring{\bar{c}}; \qquad \Pi_{\bar{c}} := \frac{\overrightarrow{\partial}}{\partial(\partial_0\bar{c})}\,\mathcal{L}_{\text{BRST}} = \mathring{c} \tag{4.8}$$

implying that the variable canonically conjugate to c is $(\partial_0\bar{c})$ and the variable conjugate to $\bar{c}$ is $(\partial_0 c)$. To write the Hamiltonian density from the Lagrangian density in the usual manner, we remember that the former has to be Hermitian, so that

$$\begin{aligned}\mathcal{H}_{\text{BRST}} &= [\Pi_k\mathring{\sigma}_k + p_\lambda\mathring{\lambda} + \Pi_0\mathring{A}_0 + E\mathring{A}_1 + \Pi_u\mathring{u} + \Pi_v\mathring{v} + \Pi_c\mathring{c} + \mathring{\bar{c}}\,\Pi_{\bar{c}} - \mathcal{L}_{\text{BRST}}] \\ &= [p_\lambda\mathring{\lambda} + \tfrac{1}{2}(\Pi_k^2 + E^2 + \sigma_k'^2 + e^2A_0^2) - \lambda(\sigma_k^2 - 1) + EA_0' + eA_0\Pi_k \\ &\quad - eA_1\sigma_k' - \tfrac{1}{2}e^2(A_0^2 - A_1^2) - \tfrac{1}{e}\Pi_0\sigma_k - \tfrac{1}{2}\Pi_0^2 + \Pi_c\Pi_{\bar{c}} + \bar{c}c]\end{aligned} \tag{4.9}$$

We can check the consistency of (4.8) and (4.9) by looking at the Hamilton equations for the fermionic variables, i.e.,

$$\partial_0 c = \frac{\vec{\partial}}{\partial \Pi_c} \mathcal{H}_{\text{BRST}}; \qquad \partial_0 \bar{c} = \mathcal{H}_{\text{BRST}} \frac{\vec{\partial}}{\partial \Pi_{\bar{c}}} \tag{4.10}$$

Thus we see that

$$\partial_0 c = \frac{\vec{\partial}}{\partial \Pi_c} \mathcal{H}_{\text{BRST}} = \Pi_{\bar{c}}; \qquad \partial_0 \bar{c} = \mathcal{H}_{\text{BRST}} \frac{\vec{\partial}}{\partial \Pi_{\bar{c}}} = \Pi_c \tag{4.11}$$

are in agreement with (4.8). For the operators c, $\bar{c}$, $\partial_0 c$, and $\partial_0 \bar{c}$, one needs to satisfy the anticommutation relations of $\partial_0 c$ with $\bar{c}$ or of $\partial_0 \bar{c}$ with c, but not of c with $\bar{c}$. In general, c and $\bar{c}$ are independent canonical variables and one assumes that

$$\{\Pi_c, \Pi_{\bar{c}}\} = \{\bar{c}, c\} = 0; \qquad \partial_0 \{\bar{c}, c\} = 0 \tag{4.12a}$$

$$\{\partial_0 \bar{c}, c\} = (-1)\{\partial_0 c, \bar{c}\} \tag{4.12b}$$

where $\{\cdot, \cdot\}$ means an anticommutator. We thus see that the anticommulators in (4.12b) are nontrivial and need to be fixed. In order to fix these, we demand that c satisfy the Heisenberg equation [3, 16–27]

$$[c, \mathcal{H}_{\text{BRST}}] = i\, \partial_0 c \tag{4.13}$$

and using the property $c^2 = \bar{c}^2 = 0$, we obtain

$$[c, \mathcal{H}_{\text{BRST}}] = \{\partial_0 \bar{c}, c\}\, \partial_0 c \tag{4.14}$$

Equations (4.12)–(4.14) then imply

$$\{\partial_0 \bar{c}, c\} = (-1)\{\partial_0 c, \bar{c}\} = i \tag{4.15}$$

Here the minus sign in the above equation is nontrivial and implies the existence of states with negative norm in the space of state vectors of the theory [3, 16–27].

4.3. The BRST Charge Operator

The BRST charge operator Q is the generator of the BRST transformations (4.1). It is nilpotent and satisfies $Q^2 = 0$. It mixes operators which satisfy Bose and Fermi statistics. According to its conventional definition, its commutators with Bose operators and its anticommutators with Fermi operators for the present theory satisfy

$$[\sigma_k, Q] = = c(e - 2\sigma_k); \qquad [\Pi_k, Q] = 2c(\sigma_k + \Pi_k + eA_0) \tag{4.16a}$$

$$[\lambda, Q] = \mathring{c}; \qquad [A_1, Q] = -c'; \qquad [A_0, Q] = \mathring{c} \tag{4.16b}$$

$$[\Pi_0, Q] = c[2e\sigma_k]; \qquad \{\bar{c}, Q\} = (\Pi_0 + p_\lambda) \tag{4.16c}$$

$$\{\mathring{\bar{c}}, Q\} = [e\Pi_k - E' - (\sigma_k^2 - 1) - 2\sigma_k\Pi_k - 2eA_0\sigma_k] \tag{4.16d}$$

All other commutators and anticommutators involving Q vanish. In view of (4.16), the BRST charge operator of the present theory can be written as

$$Q = \int dx\, \{ic[E' - e\Pi_k + (\sigma_k^2 - 1) + 2\sigma_k\Pi_k + 2eA_0\sigma_k] \\ - i\mathring{c}\, [\Pi_0 + p_\lambda]\} \tag{4.17}$$

This equation implies that the set of states satisfying the conditions

$$\Pi_0|\psi\rangle = 0 \tag{4.18a}$$

$$p_\lambda|\psi\rangle = 0 \tag{4.18b}$$

$$[E' - e\Pi_k]|\psi\rangle = 0 \tag{4.18c}$$

$$[\sigma_k^2 - 1]|\psi\rangle = 0 \tag{4.18d}$$

$$[2\sigma_k\Pi_k + 2eA_0\sigma_k]|\psi\rangle = 0 \tag{4.18e}$$

belongs to the dynamically stable subspace of states $|\psi\rangle$ satisfying $Q|\psi\rangle = 0$, i.e., it belongs to the set of BRST-invariant states.

In order to understand the condition needed for recovering the physical states of the theory, we rewrite the operators c and $\bar{c}$ in terms of fermionic annihilation and creation operators. For this purpose, we consider (4.6). The solution of eq. (4.6) gives the Heisenberg operator $c(t)$ [and correspondingly $\bar{c}(t)$] as

$$c(t) = e^{it}B + e^{-it}D; \qquad \bar{c}(t) = e^{-it}B^\dagger + e^{it}D^\dagger \tag{4.19}$$

which at time $t = 0$ imply

$$c \equiv c(0) = B + D; \qquad \bar{c} \equiv \bar{c}(0) = B^\dagger + D^\dagger \tag{4.20a}$$

$$\mathring{c} \equiv \mathring{c}(0) = i(B - D); \quad \mathring{\bar{c}} \equiv \mathring{\bar{c}}(0) = -i(B^\dagger - D^\dagger) \tag{4.20b}$$

By imposing the conditions

$$c^2 = \bar{c}^2 = \{\bar{c}, c\} = \{\mathring{\bar{c}}, \mathring{c}\} = 0 \tag{4.21a}$$

$$\{\mathring{\bar{c}}, c\} = i = -\{\mathring{c}, \bar{c}\} \tag{4.21b}$$

we obtain

$$B^2 + \{B, D\} + D^2 = B^{\dagger 2} + \{B^\dagger, D^\dagger\} + D^{\dagger 2} = 0 \tag{4.22a}$$

$$\{B, B^\dagger\} + \{D, D^\dagger\} + \{B, D^\dagger\} + \{B^\dagger, D\} = 0 \tag{4.22b}$$

$$\{B, B^\dagger\} + \{D, D^\dagger\} - \{B, D^\dagger\} - \{B^\dagger, D\} = 0 \tag{4.22c}$$

$$\{B, B^\dagger\} - \{D, D^\dagger\} - \{B, D^\dagger\} + \{D, B^\dagger\} = -1 \tag{4.22d}$$

$$\{B, B^\dagger\} - \{D, D^\dagger\} + \{B, D^\dagger\} - \{D, B^\dagger\} = -1 \tag{4.22e}$$

with the solution

$$B^2 = D^2 = B^{\dagger 2} = D^{\dagger 2} = 0 \tag{4.23a}$$

$$\{B, D\} = \{B^\dagger, D\} = \{B, D^\dagger\} = \{B^\dagger, D^\dagger\} = 0 \tag{4.23b}$$

$$\{B^\dagger, B\} = -\tfrac{1}{2}; \qquad \{D^\dagger, D\} = \tfrac{1}{2} \tag{4.23c}$$

We now let $|0\rangle$ denote the fermionic vacuum for which

$$B|0\rangle = D|0\rangle = 0 \tag{4.24}$$

Defining $|0\rangle$ to have norm one, we have that (4.23c) implies

$$\langle 0|BB^\dagger|0\rangle = -\tfrac{1}{2}; \qquad \langle 0|DD^\dagger|0\rangle = +\tfrac{1}{2} \tag{4.25}$$

so that

$$B^\dagger|0\rangle \neq 0; \qquad D^\dagger|0\rangle \neq 0 \tag{4.26}$$

The theory is thus seen to possess negative norm states in the fermionic sector. The existence of these negative norm states as free states of the fermionic part of $\mathcal{H}_{\mathrm{BRST}}$ is, however, irrelevant to the existence of physical states in the orthogonal subspace of the Hilbert space.

In terms of annihilation and creation operators,

$$\mathcal{H}_{\mathrm{BRST}} = [p_\lambda \mathring{\lambda} + \tfrac{1}{2}(\Pi_k^2 + E^2 + \sigma_k'^2 + e^2 A_0^2) - \lambda(\sigma_k^2 - 1) + EA_0' + eA_0\Pi_k$$
$$- eA_1\sigma_k' - \tfrac{1}{2}e^2(A_0^2 - A_1^2) - \tfrac{1}{e}\Pi_0\sigma_k - \tfrac{1}{2}\Pi_0^2 + 2(B^\dagger B + D^\dagger D)] \tag{4.27}$$

and the BRST charge operator Q is

$$Q = \int dx\, i\{B[E' - e\Pi_k + (\sigma_k^2 - 1) + 2\sigma_k\Pi_k + 2eA_0\sigma_k - i(\Pi_0 + p_\lambda)]$$
$$+ D[E' - e\Pi_k + (\sigma_k^2 - 1) + 2\sigma_k\Pi_k + 2eA_0\sigma_k + i(\Pi_0 + p_\lambda)]\} \tag{4.28}$$

Now, because $Q|\psi\rangle = 0$, the set of states annihilated by Q contains not only the set of states for which (4.18) hold, but also additional states for which

$$B|\psi\rangle = D|\psi\rangle = 0 \tag{4.29a}$$

$$\Pi_0|\psi\rangle \neq 0 \tag{4.29b}$$

$$p_\lambda|\psi\rangle \neq 0 \tag{4.29c}$$

$$[E' - e\Pi_k]|\psi\rangle \neq 0 \tag{4.29d}$$

$$[\sigma_k^2 - 1]|\psi\rangle \neq 0 \tag{4.29e}$$

$$[2\sigma_k\Pi_k + 2eA_0\sigma_k]|\psi\rangle \neq 0 \tag{4.29f}$$

The Hamiltonian is also invariant under the anti-BRST transformation given by

$$\bar{\hat{\delta}}\sigma_k = -e\bar{c}; \quad \bar{\hat{\delta}}A_0 = -\mathring{\bar{c}}; \quad \bar{\hat{\delta}}A_1 = -\bar{c}'; \quad \bar{\hat{\delta}}\lambda = \mathring{\bar{c}}; \quad \bar{\hat{\delta}}u = -\partial_0\partial_0 c \tag{4.30a}$$

$$\bar{\hat{\delta}}v = \partial_0\partial_0 c; \qquad \bar{\hat{\delta}}\Pi_k = \bar{\hat{\delta}}E = \bar{\hat{\delta}}\Pi_0 = \bar{\hat{\delta}}p_\lambda = \bar{\hat{\delta}}\Pi_u = \bar{\hat{\delta}}\Pi_v = 0 \tag{4.30b}$$

$$\bar{\hat{\delta}}\bar{c} = 0; \qquad \bar{\hat{\delta}}c = -b; \qquad \bar{\hat{\delta}}b = 0 \tag{4.30c}$$

with the generator or anti-BRST charge

$$\overline{Q} = \int dx\, \{(-i)c[E' - e\Pi_k + (\sigma_k^2 - 1) + 2\sigma_k\Pi_k + 2eA_0\sigma_k] + i\mathring{\bar{c}}[\Pi_0 + p_\lambda]\} \tag{4.31a}$$

$$= \int dx\,(-i)\,\{B[E' - e\Pi_k + (\sigma_k^2 - 1) + 2\sigma_k\Pi_k + 2eA_0\,\sigma_k + i(\Pi_0 + p_\lambda)] + D[E' - e\Pi_k + (\sigma_k^2 - 1) + 2\sigma_k\Pi_k + 2eA_0\sigma_k - i(\Pi_0 + p_\lambda)]\} \tag{4.31b}$$

we also have

$$\partial_0 Q = [Q, H_{\mathrm{BRST}}] = 0 \tag{4.32a}$$

$$\partial_0 \overline{Q} = [\overline{Q}, H_{\mathrm{BRST}}] = 0 \tag{4.32b}$$

with

$$H_{\mathrm{BRST}} = \int dx\, \mathcal{H}_{\mathrm{BRST}} \tag{4.32c}$$

and we further impose the dual condition that both Q and $\overline{Q}$ annihilate physical states, implying that

$$Q|\psi\rangle = 0 \qquad \text{and} \qquad \overline{Q}|\psi\rangle = 0 \tag{4.33}$$

The states for which (4.18) hold satisfy both of these conditions and, in fact, are the only states satisfying both of these conditions since, although with (4.23)

$$2(B^\dagger B + D^\dagger D) = -2(BB^\dagger + DD^\dagger) \tag{4.34}$$

there are no states of this operator with $B^\dagger|0\rangle = 0$ and $D^\dagger|0\rangle = 0$ [cf. (4.26)], and hence no free eigenstates of the fermionic part of H_{BRST} which are annihilated by each of B, $B^{\dagger}$, D, and $D^{\dagger}$. Thus the only states satisfying (4.33) are those satisfying the constraints (3.12).

Further, the states for which (4.18) holds satisfy both the conditions (4.33) and in fact are the only states satisfying both of these conditions because in view of (4.21), one cannot have simultaneously c, $\partial_0 c$ and $\bar{c}$, $\partial_0 \bar{c}$ applied to $|\psi\rangle$ to give zero. Thus, the only states satisfying (4.33) are those that satisfy the constraints of the theory (3.12) and they belong to the set of BRST-invariant and anti-BRST-invariant states.

Alternatively, one can understand the above point in terms of fermionic annihilation and creation operators as follows. The condition $Q|\psi\rangle = 0$ implies that the set of states annihilated by Q contains not only the states for which (4.18) holds, but also additional states for which (4.29) holds. However, $\overline{Q}|\psi\rangle = 0$ guarantees that the set of states annihilated by $\overline{Q}$ contains only the states for which (4.18) holds, simply because $B^{\dagger}|\psi\rangle \neq 0$ and $D^{\dagger}|\psi\rangle \neq 0$. Thus, in this alternative way, we also see that the states satisfying $Q|\psi\rangle = \overline{Q}|\psi> = 0$ [i.e., satisfying (4.33)] are only those states that satisfy the constraints of the theory and also that these states belong to the set of BRST-invariant and anti-BRST-invariant states.

5. SUMMARY AND DISCUSSION

In this work, we have considered a GNLSM in the instant form of dynamics on the hyperplanes $x^0 = \text{const}$. The theory is seen to possess a set of five first-class constraints Ω_i. The theory is indeed seen to possess a local vector gauge symmetry. At the classical level, the divergence of the local vector gauge current density $\partial_\mu j^\mu$ vanishes under the gauge constraint $\lambda \approx 0$, which is, in fact, equivalent to the temporal or time-axial gauge for the coordinate λ. The BRST quantization of the theory has also been studied under specific gauge choices. The gauge choices considered in the present work are, however, not unique and one could consider the theory under other gauges as well. It is important to note here that the usual NLSM (which does not have any gauge fields at all) is a GNI theory possessing a set of second-class constraints [1–4]. It is also important to note that in the present GNLSM, one does not encounter any operator ordering problem [1, 2, 4]. This is in contrast to the usual NLSM, where one does encounter the operator ordering problem [1, 2, 4].

ACKNOWLEDGMENTS

I thank Prof. A. N. Mitra, Prof. Werner Ruehl, Prof. Andreas Wipf, Prof. Martin Reuter, and Prof. D. S. Kulshreshtha for very helpful discussions, and the CSIR, New Delhi, for the award of a Senior Research Associateship which enabled her to carry out this research.

REFERENCES

1. J. Maharana (1983). *Phys. Lett.* **B 128**, 411.
2. J. Maharana (1983). *Ann. Inst. Henri Poincaré* **XXXIX**, 193, and references therein.
3. P. Mitra and R. Rajaraman (1990). *Ann. Phys.* (N.Y.) **203**, 137, 157.
4. U. Kulshreshtha, D. S. Kulshreshtha, and H. J. W. Muller-Kirsten (1993). *Helv. Phys. Acta* **66**, 752.
5. W. Ruehl (1991). *Ann. Phys.* (N.Y.) **206**, 368.
6. W. Ruehl (1996). *Ann. Phys.* (N.Y.) **247**, 414.
7. W. Ruehl, (1995). *Mod. Phys. Lett.* **A10**, 2353.
8. W. Ruehl (1991/1993). Conformal quantum field theory in two dimensions, Parts I + II, Lecture Notes, Kaiserslautern, and references therein.
9. S. Coleman, J. Wess, and B. Zumino (1969). *Phys. Rev.* **177**, 2239.
10. C. Callen, S. Coleman, J. Wess, and B. Zumino (1969). *Phys. Rev.* **177**, 2247.
11. P. Candelas, G. Horwitz, A. Strominger, and E. Witten (1985). *Nucl. Phys. B*, **258**, 46.
12. M. Henneaux and L. Mezincescu (1985). *Phys. Lett.* **B 152**, 340.
13. A. B. Zamolodchikov and A. B. Zamolodchikov (1979). *Ann. Phys.* (N. Y.) **120**, 253.
14. P. A. M. Dirac (1950). *Can. J. Math.* **2**, 129. *Lectures on Quantum Mechanics* (Yeshiva University Press, New York).
15. P. A. M. Dirac (1964). *Lectures on Quantum Mechanics* (Yeshiva University Press, New York).
16. C. Becchi, A. Rouet, and R. Stora (1974). *Phys. Lett.* **52B**, 344.
17. V. Tyutini (1975). Lebedev Report No. FIAN-39.
18. M. Henneaux and C. Teitelboim (1992). *Quantization of Gauge Systems* (Princeton University Press, Princeton, New Jersey).
19. D. Nemeschansky, C. Preitschopf, and M. Weinstein (1988). *Ann. Phys.* (N. Y.) **183**, 226.
20. U. Kulshreshtha (1998). *Helv. Phys. Acta* **71**, 353.
21. U. Kulshreshtha and D. S. Kulshreshtha (1998). *Int. J. Theor. Phys.* **37**, 2603.
22. U. Kulshreshtha, D. S. Kulshreshtha, and H. J. W. Muller-Kirsten (1993). *Helv. Phys. Acta.* **66**, 737.
23. U. Kulshreshtha, D. S. Kulshreshtha, & H. J. Muller-Kirsten (1994). *Nuovo Cimento* **107A**, 569.
24. U. Kulshreshtha, D. S. Kulshreshtha, & H. J. Muller-Kirsten (1994). *Can. J. Phys.* **72**, 639.
25. U. Kulshreshtha, D. S. Kulshreshtha, & H. J. Muller-Kirsten (1994). *Z. Phys. C* **64**, 169.
26. U. Kulshreshtha, D. S. Kulshreshtha, & H. J. Muller-Kirsten (1995). *Can. J. Phys.* **73**, 386.
27. U. Kulshreshtha, D. S. Kulshreshtha, & H. J. Muller-Kirsten (1999). *Int. J. Theor. Phys.* **38**, 1411.
28. P. A. M. Dirac (1949). *Rev. Mod. Phys.* **21**, 392.